Sparsifying, Shrinking and Splicing for Minimum Path Cover in Parameterized Linear Time

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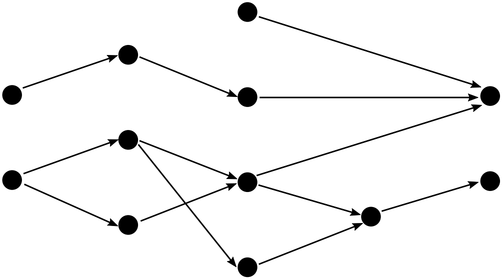
Problem & Motivation

MPC of a DAG



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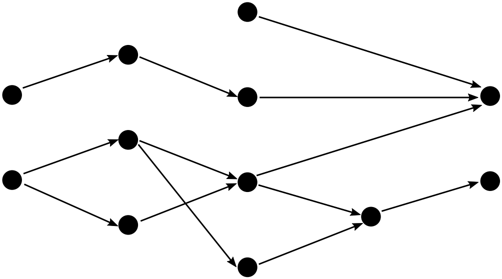

A minimum-sized set of paths such that every vertex appears in at least one path of the set

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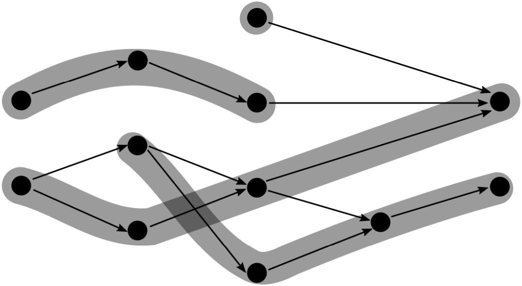

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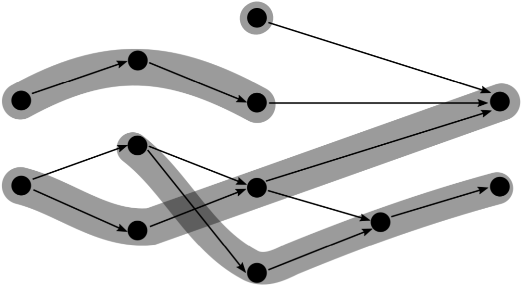

A minimum-sized set of paths such that every vertex appears in at least one path of the set

→ Solvable in polynomial time [9, 11]

Motivation

Applications in various fields

- Bioinformatics
  - Multi-assembly [10, 27, 25, 4, 19]
  - Perfect phylogeny haplotyping [1, 13]
  - Pan-genome alignment [22, 20]

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- Scheduling [7, 8, 3, 28, 29, 23], computational logic [2, 12], distributed computing [26, 15], databases [16], evolutionary computation [17], program testing [24], cryptography [21], programming languages [18]

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$\longrightarrow$Since the size k of an MPC (*width*) is small, research has focused in solutions parameterized by k

Previous work

MPC algorithms

Reduction to Maximum Matching [11]

- $O(\sqrt{|V|}|E|)$ [14] $\rightarrow$ transitive DAGs
- $O(|V|^2 + k\sqrt{k}|V|)$ and $O(\sqrt{|V|}|E| + k\sqrt{k}|V|)$ [5, 6]

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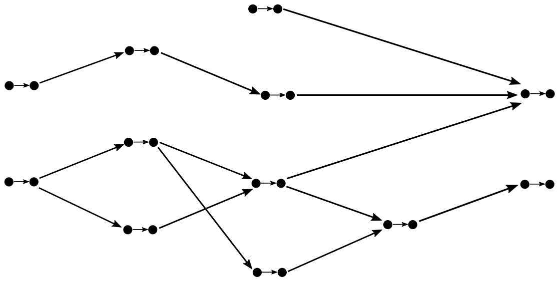

$$f(v^{in}, v^{out}) \geq d(v^{in}, v^{out}) = 1$$

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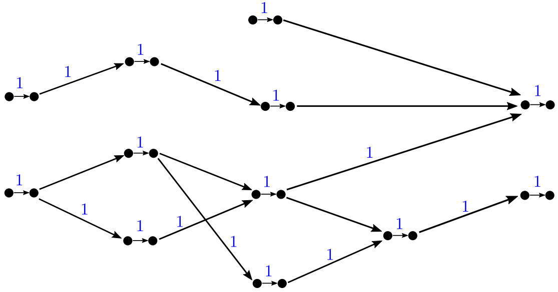

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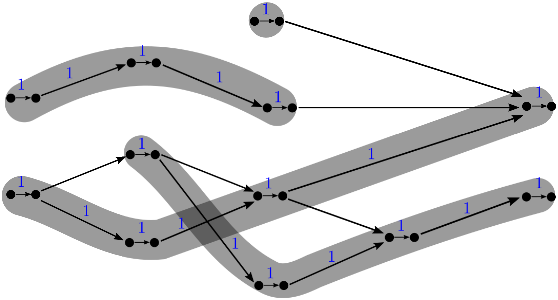

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None of them reaches parameterized linear time

Our results

MPC in parameterized linear time

- A simple D&C algorithm
  - $O(k^2|V|\log|V| + |E|)$
  - $O(k^2|V| + |E|)$ in PRAM
- The first parameterized linear time algorithm
  - $O(k^3|V| + |E|)$
- Width sparsification of edges to $< 2|V|$
  - $O(k^3|V| + |E|)$

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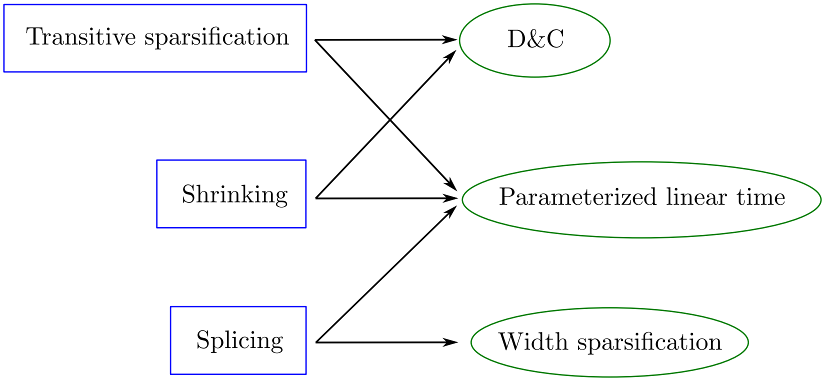
$\longrightarrow$At the core of our solutions we use:
Transitive sparsification, shrinking and splicing

Roadmap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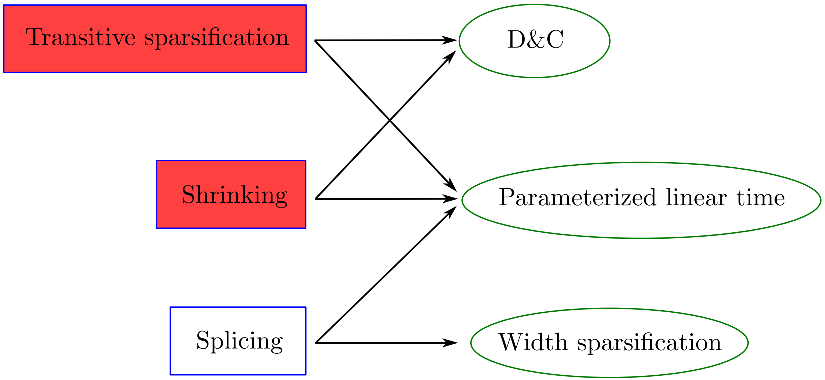

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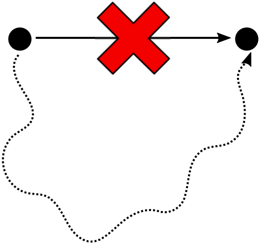
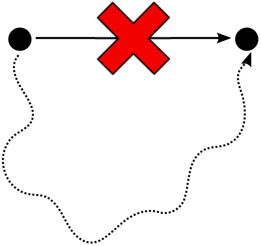


A spanning subgraph S preserving the reachability relation between its vertices

Transitive sparsification



A spanning subgraph S preserving the reachability relation between its vertices
$\Rightarrow$

- $\mathsf{width}(S) = \mathsf{width}(G)$
- Every path cover of S is path cover of G

Transitive sparsification

Inspired by Jagadish's work [16], we propose a transitive sparsification algorithm based on a path cover $\mathcal{P}$ of size t

Observation 2.1

We sparsify the incoming edges of v to $\leq t$ in $O(t + |N^{-}(v)|)$

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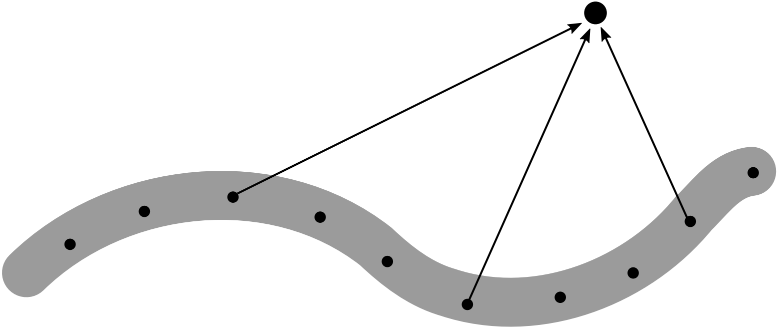

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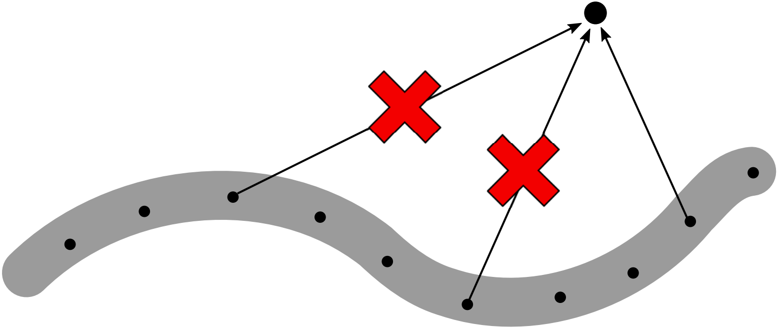

Shrinking

Shrink a path cover $\mathcal{P}$ of size t in an MPC

Can be solved by using the flow reduction:

- Interpret $\mathcal{P}$ as a flow
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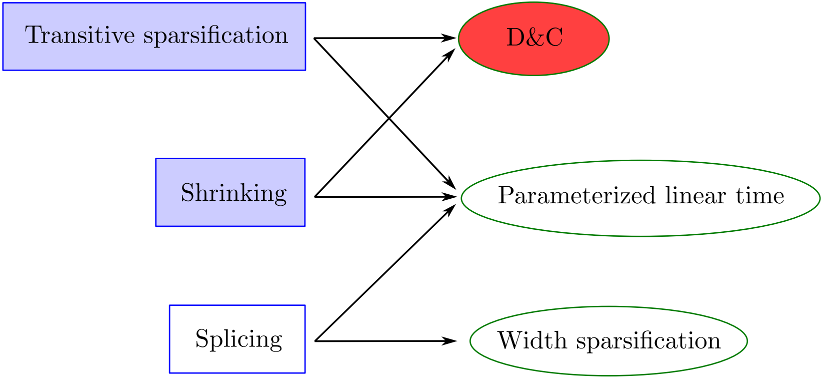
Lemma 2.5

We can obtain an MPC of G in time $O(t(|V| + |E|))$

$\longrightarrow$ Generalization of approach used by Mäkinen et.al [22]

Roadmap



Transitive sparsification
D&C
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- Compute a **topological order** of the vertices $v_1, \ldots, v_{|V|}$

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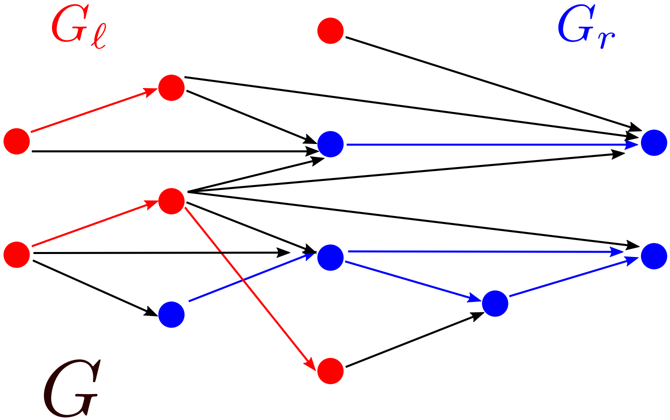
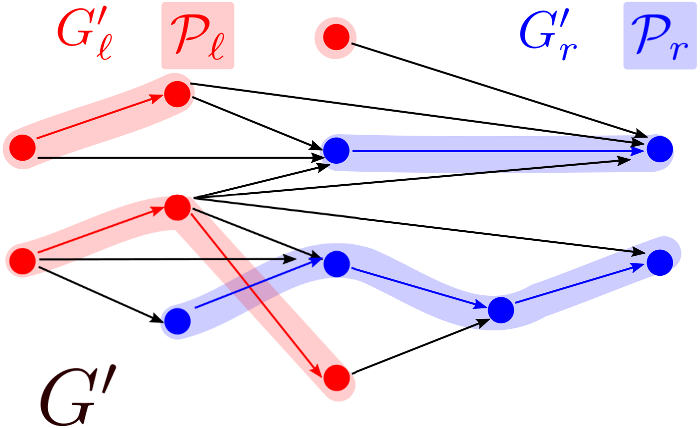
  - **Shrink** the path cover solution to $P_1, \ldots, P_k$

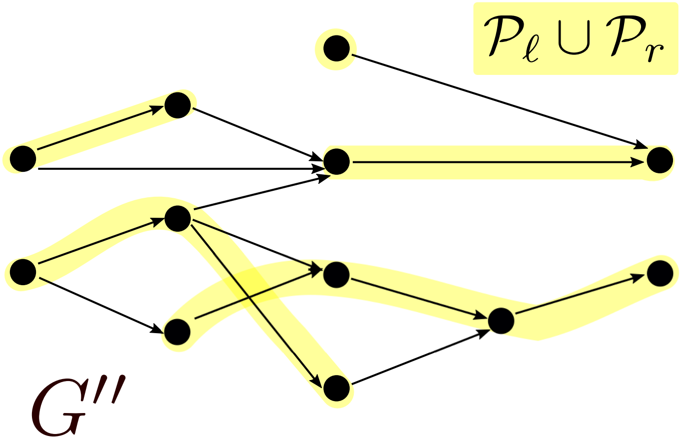
An example - Division

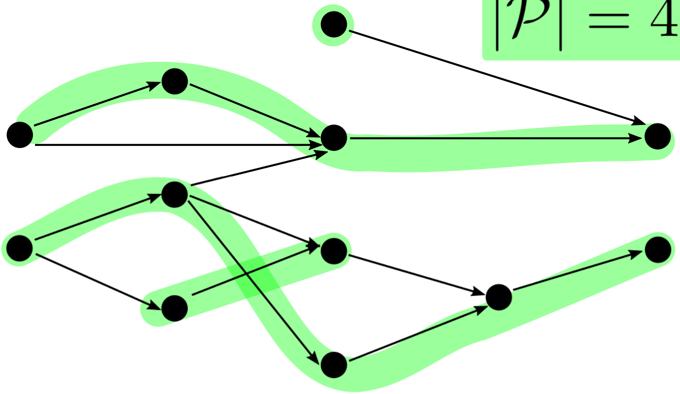


An example - Recursion



An example - Sparsification





$|\mathcal{P}| = 4$

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Theorem 1.1

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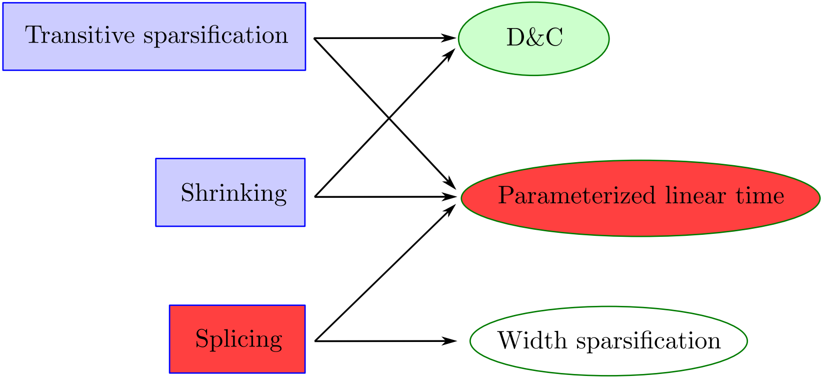
Theorem 1.2

We compute an MPC in $O(k^2|V| + |E|)$ *parallel steps using* $O(\log|V|)$ *single processors in the PRAM model*

Roadmap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Parameterized linear time
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Parameterized linear time

High-level approach:

- Process vertices in topological order $v_1, \ldots, v_{|V|}$
- At each step compute an MPC $\mathcal{P}_i$ of G_i
- In step $i+1$ consider the path cover $\mathcal{T}_{i+1} = \mathcal{P}_i \cup \{v_{i+1}\}$, and **shrink** it to $\mathcal{P}_{i+1}$

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$\longrightarrow$ We work directly with the *flow reduction* $\mathcal{G}_i$ of G_i, and at each step we look for a *decrementing path* in the residual network $\mathcal{R}(\mathcal{G}_{i+1}, \mathcal{T}_{i+1})$

Parameterized linear time

At step $i+1$:

1. **Sparsify** the edges incoming to v_{i+1}^{in} using $\mathcal{P}_i$
   - Ensures $O(k)$ out-neighbors in $\mathcal{R}(\mathcal{G}_{i+1}, \mathcal{T}_{i+1})$
2. Layered traversal of $\mathcal{R}(\mathcal{G}_{i+1}, \mathcal{T}_{i+1})$
3. If a decrementing path D is found, **splice** $\mathcal{T}_{i+1}$ along D to get $\mathcal{P}_{i+1}$. Otherwise, $\mathcal{P}_{i+1} \leftarrow \mathcal{T}_{i+1}$
4. Update level of vertices

Layered Traversal

We maintain:

- Level assignment ℓ of the vertices of $\mathcal{G}_i$ to $\{0, 1, \ldots, \mathsf{width}(G_i)\}$. Partition of vertices into *layers*
- **Invariant A**: For each (u, v) in $\mathcal{R}(\mathcal{G}_i, \mathcal{P}_i)$, $\ell(u) \geq \ell(v)$
- **Invariants B and C**

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Main idea:

BFS in each reachable layer from highest to lowest

- Stop when reaching t
- Only continues to the next highest reachable layer once all reachable vertices from the current layer have been visited

Splicing

Reconnecting paths in a path cover $\mathcal{P}$ so that one follows a certain path D

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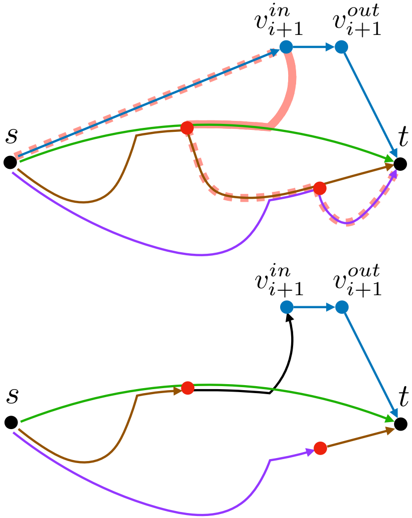
Reconnecting paths in a path cover $\mathcal{P}$ so that one follows a certain path D

- Requires edges in D covered by $\mathcal{P}$
- Preserves covering of vertices, size of path cover, and multiplicity of edges

Lemma 2.6

We can obtain, in $O(|D|)$, a path cover $\mathcal{P}'$ of the same size such that $\mu_{\mathcal{P}}(e) = \mu_{\mathcal{P}'}(e)$, and there exists $P \in \mathcal{P}'$ containing D

Splice $\mathcal{T}_{i+1}$ along D



Transform $\mathcal{T}_{i+1}$ into $\mathcal{P}_{i+1}$ in $O(|D|)$

Level update

If l is the smallest layer visited by the traversal, we set:

- $\ell(v_{i+1}^{in}) = l$, $\ell(v_{i+1}^{out}) = l + 1$
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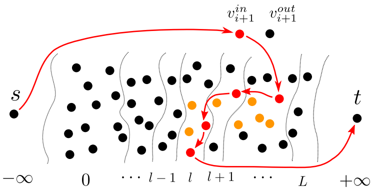

Parameterized linear time

We show that:

- Invariants are maintained
- A step takes $O(|N^-(v_{i+1})|)$ and $O(k)$ per vertex of level at least l
- Each vertex is charged $O(k^2)$ times during the algorithm

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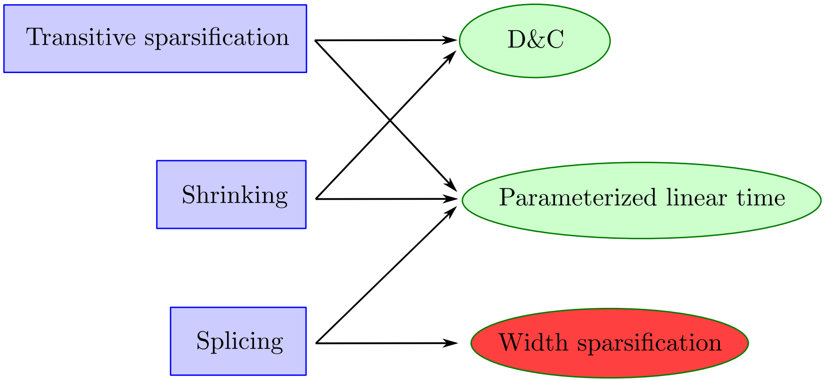
Theorem 1.3

We compute an MPC in time $O(k^3|V| + |E|)$

Roadmap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Width sparsification

We show the following result for a path cover of size t

Theorem 1.4

We compute, in $O(t^2|V|)$ time, a path cover $\mathcal{P}', |\mathcal{P}'| = t$, whose number of distinct edges is less than $2|V|$

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Thus we obtain

Corollary 1.1

We compute a spanning subgraph $G' = (V, E')$ of G with $|E'| < 2|V|$ and width k in time $O(k^3|V| + |E|)$.

Width sparsification

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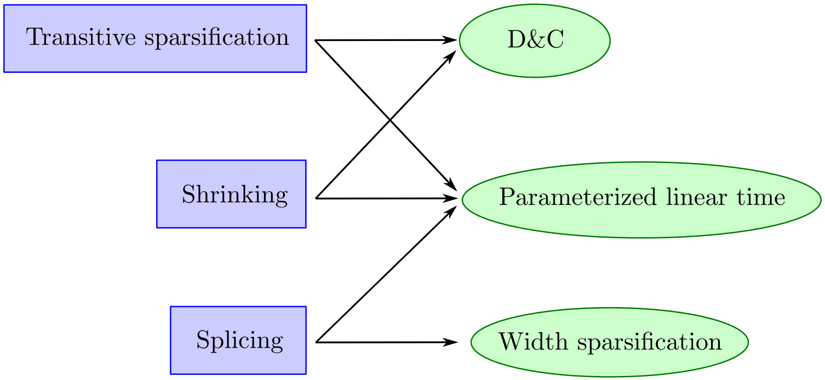
$\longrightarrow$ We also show that the bound $2|V|$ is asymptotically tight

Roadmap



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Funding

European Research Council

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